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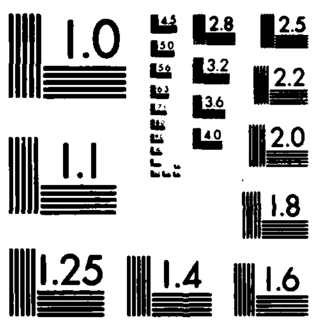

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RAY METHOD FOR FLOW OF A COMPRESSIBLE VISCOUS FLUID

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ABSTRACT

An asymptotic method is developed for the solution of linearized equations governing the flow of a compressible viscous fluid with a free surface. The approach used here is based upon the ray method expansion originally developed by Keller. A general uniform asymptotic expansion is also constructed to remove anomalies in which an amplitude function in the ray method expansion becomes infinite.

AMS (MOS) Subject Classifications: 76N10, 78A05

Key Words: Compressible viscous flow, free surface, ray method, uniform asymptotic expansion.

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SIGNIFICANCE AND EXPLANATION

The authors describe

~~We develop~~ an asymptotic method to solve the linearized Navier-Stokes equations governing the flow of a compressible viscous fluid subject to free surface and rigid bottom boundary conditions. The solution of these equations is assumed to consist of a phase function and an amplitude function. It is found that the phase function satisfies the Hamilton-Jacobi equation, and the first order approximation to the amplitude function satisfies a transport equation. The Hamilton-Jacobi equation may be solved by means of the method of characteristics, which reduces the equation to a set of ordinary differential equations. Their solutions determine a family of time-space curves called rays. The transport equation can be easily integrated along each ray to yield the so-called conservation relation. At certain anomalies the amplitude function becomes infinite and a uniform expansion is then constructed to remove these difficulties.

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The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

RAY METHOD FOR FLOW OF A COMPRESSIBLE VISCOUS FLUID

M. C. Shen*

I. Introduction

Since the ray method was systematically developed by Keller and his co-workers, much progress has been made in recent years and a review of the method can be found in Keller [1]. Although the ray method was first formulated for electromagnetic waves and has been extended to many other related areas, it has also found a lot of interesting and fruitful applications to the wave propagation problems of a fluid, especially with a free surface. In the past decade, many results have been obtained for the case of an irrotational, inviscid or incompressible, viscous fluid with free surface. However, only recently the Navier-Stokes equations governing the motion of a compressible viscous fluid have received much attention, as shown in the review article by Solonnikov and Kazhiklov [2]. At present very few solutions of the linearized Navier-Stokes equations subject to free surface conditions are available, to say nothing of the nonlinear ones. It should be of great interest to develop an asymptotic method to study these equations. Indeed their physical applications abound, for example, wave propagation in the atmosphere or ocean. A compressible viscous fluid may be also used to model flow in porous media, a discussion of which may be found in Bear [3].

In the derivation of the ray method by Keller [1] for surface waves on water over a variable bottom, upon which our approach is based, a wave-like solution with an amplitude function and a phase function was assumed, and the

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boundary conditions at the free surface and the rigid bottom were taken into account as an integral part of the formulation of the method. The three-dimensional problem is reduced to the problem of constructing rays and wave fronts on the two-dimensional equilibrium free surface, and the solution of partial differential equations is reduced to the integration of ordinary differential equations along rays. This method has since proven very successful in dealing with the problems of wave propagation in a fluid with a free surface (Shen, [5]). For the problem considered here, we adopt the form for the ray method expansion used in Cohen and Lewis [6], Voronka and Keller [7]. It is also a well known fact that the ray method expansion in its usual form may fail at anomalies where the amplitude function becomes infinite. We then use a modified version of the uniform asymptotic expansion in Shen and Keller [8] to remove these difficulties. The uniform asyptotic expansion at a caustic was first developed by Ludwig [9] and Krautsov [10].

The approach used here may be sketched as follows: We consider a shallow layer of a compressible viscous fluid supported below by a rigid bottom and bounded above by a free surface. The rigid bottom is assumed to vary slowly in two horizontal directions. A large parameter β is introduced as the ratio of the horizontal length scale L to the vertical length scale H. The governing equations are nondimensionalized in terms of the dimensionless variables appropriately chosen. Furthermore, we also assume that the constant viscosity is of small order so that a Reynolds number to be defined is large. First we construct the usual ray method expansion involving a phase function and an amplitude function. By substituting the expansion in the equations, we obtain the Hamilton-Jacobi equation for the phase function, which in turn determines a family of bicharacteristics called rays. The successive approximations to the amplitude function may be obtained by integrating transport equations along the ray. The construction of a uniform asymptotic expansion is motivated by the form used in [9], [10], and based upon an ordinary differential equation as a comparison equation. A transformation is discovered to reduce the results for the ray method expansion to those for the uniform asumptotic expansion. By appropriately chosing the coefficients in the comparison equation and a conservation relation along each ray the amplitude function becomes finite at an anomally.

We formulate our problem and construct the ray method expansion in §2. The uniform asymptotic expansion is given in §3. Discussions about some special cases and possible extension of the results obtained are presented in §4. Some of the detailed derivations are deferred to the appendices.

2. Ray Method Expansion

The linearized Navier-Stokes equations governing the motion of a compressible viscous fluid are assumed to be the following:

$$\rho^*_{t^*} + (\rho_0^* u^*)_{x^*} + (\rho_0^* v^*)_{y^*} + (\rho_0^* w^*)_{z^*} = 0 \quad , \tag{1}$$

$$\rho_0^* u^*_{t^*} = -p^*_{x^*} + \mu\, \nabla_*^2 u^* + (\mu/3)(\nabla_* \cdot \vec{u}^*)_{x^*} \quad , \tag{2}$$

$$\rho_0^* v^*_{t^*} = -p^*_{y^*} + \mu\, \nabla_*^2 v^* + (\mu/3)(\nabla_* \cdot \vec{u}^*)_{y^*} \quad , \tag{3}$$

$$\rho_0^* w^*_{t^*} = -p^*_{z^*} - \rho^* g + \mu\, \nabla_*^2 w^* + (\mu/3)(\nabla_* \cdot \vec{u}^*)_{z^*} \quad , \tag{4}$$

$$\rho^* = (d\rho_0^*/dp_0^*)p^* \quad , \tag{5}$$

subject to the boundary conditions:

At $z = 0$,

$$u^*_{z^*} + w^*_{z^*} = 0 \quad , \tag{6}$$

$$v^*_{z^*} + w^*_{y^*} = 0 \quad , \tag{7}$$

$$-p^* + \rho_0^* g\eta^* - (2\mu/3)\nabla_* \cdot \vec{u}^* + 2\mu w_z = 0 \quad , \tag{8}$$

$$\eta^*_{t^*} - w^* = 0 \quad ; \tag{9}$$

$$\text{at} \quad z^* = -h^*(x^*,y^*), \quad \vec{u}^* = 0 \quad . \tag{10}$$

Here x^*,y^* are horizontal coordinates and z^* is positive upward, t^* is the time, $\nabla_* = (\partial/_{\partial x^*}, \partial/_{\partial y^*}, \partial/_{\partial z^*})$, $\vec{u}^* = (u^*,v^*,w^*)$ is the velocity, ρ_0^* is the equilibrium density, considered as a function of p_0^*, the equilibrium pressure, ρ^* is the density, g^* is the constant gravitational acceleration, μ is the constant viscosity, $z^* = \eta^*$ is the equation of the free surface and $z^* = -h^*$ is the equation of the rigid bottom assumed to vary slowly in the x^* and y^* directions. We note that p_0^* is a function of z^* satisfying $p_{0_{z^*}}^* = -\rho_0^*$, $p_0^* = 0$ at $z^* = 0$.

Motivated by the assumptions made before, we introduce nondimensional variables and parameters as below:

$$(x,y,z) = (\beta^{-2}x^*,\beta^{-2}y^*,z^*)/H,\ t = \beta^{-2}t^*/(H/g)^{1/2}\ ,$$

$$\bar{q} = (u,v) = (u^*,v^*)/(gH)^{1/2},\ w = \beta w^*/(gH)^{1/2}\ ,$$

$$p = p^*/[\rho_0^*(0)gH],\ p_0 = p_0^*/[\rho_0^*(0)gH],\ \rho = \rho^*/\rho_0^*(0)\ ,$$

$$\rho_0^* = \rho_0/\rho_0^*(0),\ \eta = \eta^*/H,\ h = h^*/H\ ,$$

$$R = \rho_0^*(0)(gH)^{1/2}H/\mu,\ \beta = L/H\ ,$$

where we assume R is large and without loss of generality we set $R = \beta$. In terms of the nondimensional variables (1) to (10) become

$$\rho_t + \nabla\cdot(\rho_0\bar{q}) + \beta(\rho_0 W)_z = 0\ , \tag{11}$$

$$\rho_0\bar{q}_t + \nabla p = \beta^{-3}\nabla^2\bar{q} + \beta\bar{q}_{zz} + (1/3)\nabla(\nabla\cdot\bar{q}) + (1/3)\beta^{-1}\nabla W_z\ , \tag{12}$$

$$\rho_0 W_t + \beta^3(p_3+\rho) = \beta^{-3}\nabla^2 W + \beta W_{zz} + (1/3)(\nabla\cdot\bar{q})_z + (\beta/3)W_{zz}\ , \tag{13}$$

$$\rho = \rho_0'(p_0)p\ ; \tag{14}$$

at $z = 0$,

$$\beta^3\bar{q}_z + \nabla W = 0\ , \tag{15}$$

$$\beta^2(p-\eta) + (2/3)(\beta^{-1}\nabla\cdot\bar{q} + W_z) - 2w_z = 0\ , \tag{16}$$

$$\eta_t - \beta W = 0\ ; \tag{17}$$

at $z = -h$,

$$\bar{q} = 0,\ W = 0\ , \tag{18}$$

where $$\nabla = (\partial/\partial x,\ \partial/\partial y)\ .$$

For large β, we assume that the solutions $\bar{q}$, W, p, ρ and η possess an expansion of the form

$$\phi \sim \exp[-\beta S(t,x,y)](\phi_1 + \beta^{-1}\phi_2 + \beta^{-2}\phi_3 + \cdots) \quad , \tag{19}$$

where S is called the phase function. Substituting (19) in (1) to (18), we obtain a sequence of equations and boundary conditions for successive approximations to the amplitude function. The equations for the first approximation are

$$\omega\rho_1 - \rho_0\bar{k} \cdot \bar{q} + (\rho_0 W_1)_z = 0 \quad , \tag{20}$$

$$\omega\rho_0\bar{q}_1 - \bar{k}p_1 = \bar{q}_{1zz} \quad , \tag{21}$$

$$p_{1z} = -\rho_1 \quad , \tag{22}$$

$$\rho_1 = \rho_0^1(p_0)p_1 \quad ; \tag{23}$$

at $z = 0$,

$$\bar{q}_{1z} = 0, \; p_1 = \eta_1, \; \omega\eta_1 - W_1 = 0 \quad ; \tag{24}$$

at $z = -h$,

$$\bar{q}_1 = 0, \; W_1 = 0 \quad , \tag{25}$$

where $$\omega = -S_t, \; \bar{k} = \nabla S \quad .$$

From (22) to (24), it is obtained that

$$p_1 = \eta_1 \exp[-\int_0^z \rho_0^1(p_0)dz] = \eta_1\rho_0, \; \rho_1 = \rho_0^1\rho_0\eta_1 \quad , \tag{26}$$

where $p_{0z} = -\rho_0$, $\rho_0(0) = 1$ have been used. Now let

$$\bar{q} = \bar{k}\eta_1\Omega_1 \quad , \tag{27}$$

and it follows from (21), (24) and (25) that

$$\Omega_{1zz} - \rho_0\omega\Omega_1 = -\rho_0 \quad , \tag{28}$$

$$\Omega_{1z} = 0 \quad \text{at} \quad z = 0 \quad , \tag{29}$$

$$\Omega_1 = 0 \quad \text{at} \quad z = -h \quad . \tag{30}$$

Assume that the Green's function of (28) to (30) is given by $G(z,z',\omega,h)$. Then

$$\Omega_1 = -\int_{-h}^{0} \rho_0 G(z,z',\omega,h)dz' \quad . \tag{31}$$

Finally, we integrate (20) and make use of (25) to obtain

$$w_1 = \omega\eta_1(\rho_0 - \rho_0(-h)) + k^2\eta_1 \int_{-h}^{3} \rho_0\Omega_1 dz \quad , \tag{32}$$

where $k = |\bar{k}|$. However, w_1 must also satisfy (24) and it follows that, assuming $\eta_1 \neq 0$,

$$\omega\rho_0(-h) = k^2 \int_{-h}^{0} \rho_0\Omega_1 dz, \quad \text{or} \quad k^2 = H(\omega,h) \quad , \tag{33}$$

which yields an equation in the form of the Hamilton-Jacobi equation, and may be solved by the method of Characteristics. The corresponding characteristic equations are

$$\begin{aligned} dt/d\sigma &= \mu H_\omega, \quad d\bar{r}/d\sigma = 2\mu\bar{k}, \quad d\omega/d\sigma = 0 \quad , \\ d\bar{k}/d\sigma &= -\mu\nabla H, \quad dS/d\sigma = \mu(2k^2 - \omega) \quad , \end{aligned} \tag{34}$$

where μ is a proportionality factor and $\bar{r} = (x,y)$. The solutions of (34) determine a two-parameter family of bicharacteristics

$$t = t(\sigma,\sigma_1,\sigma_2), \quad \bar{r} = \bar{r}(\sigma,\sigma_1,\sigma_2) \quad , \tag{35}$$

which will be called rays. From (34) we also observe that ω is constant along a ray, and the phase function is determined by

$$S = S(\sigma_0) + \int_{\sigma_0}^{\sigma} \mu(2k^2 - \omega)d\sigma \quad . \tag{36}$$

where $S(\sigma_0)$ is the value of S at some point $\sigma = \sigma_0$ on a ray.

To determine η_1 we proceed to the equations for the second approximation

$$\omega\rho_2 + \rho_1 t - \rho_0\bar{k} \cdot \bar{q}_2 + \nabla\cdot(\rho_0\bar{q}_2) + (\rho_0 W_2)_z = 0 \quad , \tag{37}$$

$$\omega\rho_0\bar{q}_2 + \rho_0\bar{q}_{1t} - \bar{k}p_2 + \nabla p_1 = \bar{q}_{2zz} - (1/3)\bar{k}W_{1z} \quad , \tag{38}$$

$$p_{2z} = -\rho_2 \quad , \tag{39}$$

$$\rho_2 = \rho_0'(p_0)p_2 \quad , \tag{40}$$

at $z = 0$,

$$\bar{q}_{2z} = 0, \; p_2 = \eta_2, \; \omega\eta_2 + \eta_{1t} - W_2 = 0 \quad , \tag{41}$$

at $= -h$,

$$\bar{q}_2 = 0, \quad W_2 = 0 \quad . \tag{42}$$

From (39) to (41), we obta n as before that

$$p_2 = \rho_0 \eta_2, \; \rho_2 = \rho_0'(p_0)\rho_0\eta_2 \quad . \tag{43}$$

Now we let

$$\bar{i}_2 = (\bar{k}\eta_2 - \nabla\eta_1)Q_1 + \bar{Q}_2 \quad , \tag{44}$$

where Q_1 satisfies (28) to (30). Then it follows from (38), (41) and (42) that

$$\bar{Q}_{2zz} - \omega\rho_0\bar{Q}_2 = \rho_0\bar{q}_{1t} + (1/3)\bar{k}W_{1z} \quad , \tag{45}$$

$$\bar{Q}_{2z} = 0, \quad \text{at} \quad z = 0 \quad , \tag{46}$$

$$\bar{Q}_2 = 0, \quad \text{at} \quad z = -h \quad . \tag{47}$$

Integrating (37) with respect to z from $z = -h$ to $z = 0$ and making use of (27), (33), (41) and (42), we have

$$\int_{-h}^{0} [\eta_{1t}\rho_0'(p_0)\rho_0 + \rho_0 Q_1\bar{k} \cdot \nabla\eta_1 - \rho_0\bar{k} \cdot \bar{Q}_2 + \nabla\cdot(\rho_0\bar{k}\eta_1 Q_1)]dz + \eta_{1t} = 0 \quad . \tag{48}$$

Upon multiplying (48) by $2\eta_1$ and rearranging the terms, we finally obtain

$$(I\eta_1^2)_t + \nabla\cdot(I\eta_1^2\, d\bar{r}/dt) - A\eta_1^2 = 0 \quad , \tag{49}$$

where

$$I = \rho_0(-h) - k^2 \int_{-h}^{0} \rho_0 Q_1 \omega \, dz \quad ,$$

$$A = k^2 \int_{-h}^{0} \rho_0(z) \int_{-h}^{0} G(z,z',\omega,h)[\omega\rho_0 z' + k^2 Q_1(z')\rho_0(z')]dz' \quad .$$

The detailed drivation of (49) is deferred to Appendix A. Let $J(\sigma)$ be the Jacobian of transformation from the ray coordinates σ, σ_1, σ_2 to the t, x, y coordinates, where we choose $t = \sigma$. By the well-known identity

$$J^{-1}\, dJ/dt = \nabla\cdot(d\bar{r}/dt) \quad ,$$

(49) reduces to

$$d(IJ\eta_1^2)/dt = AJ\eta_1^2 \quad . \tag{50}$$

Integration of (50) along a ray yields

$$J I \eta_1^2 \exp[-\int_{t_0}^{t} A I^{-1} dt] = \text{constant} \quad . \tag{51}$$

As seen from (51) η_1 may become infinite at anomalies where, for example, $J(t) = 0$. To remove them, we shall construct a uniform asymptotic expansion in the next section.

3. Uniform asymptotic expansion

The construction of a uniform asymptotic expansion is based upon the approach developed in [11], and a discussion of the method may be found in [8]. Assume that $\bar{q}$, w, p, ρ and η possess an asymptotic expansion of the form

$$\phi = \exp(-\beta\theta)[\phi^{(1)}V(\beta^{\nu}\xi) + \beta^{\nu-1}\phi^{(2)}V'(\beta^{\nu}\xi)] \quad , \tag{52}$$

where $V(\beta^{\nu}\xi)$ satisfies the second order ordinary differential equation

$$\begin{aligned} &V''(\beta^{\nu}\xi) + \beta^{-\nu}P(\xi)V'(\beta^{\nu}\xi) - \beta^{2-2\nu}Q^2(\xi)V(\beta^{\nu}\xi) = 0 \quad , \\ &\phi^{(i)} \sim \phi_1^{(i)} + \beta^{-1}\phi_2^{(i)} + \beta^{-2}\phi_3^{(i)} + \cdots \quad , \quad i = 1,2 \quad . \end{aligned} \tag{53}$$

θ and ξ are functions of t and $\bar{r}$ only, ν is a nonnegative number, and $P(\xi)$, $Q(\xi)$ are functions to be chosen. We define

$$\phi^{\pm} = \phi^{(1)} \pm Q\phi^{(2)} \sim \phi_1^{\pm} + \beta^{-1}\phi_2^{\pm} + \beta^{-2}\phi_3^{\pm} + \cdots \quad , \tag{54}$$

$$S^{\pm} = \theta \pm \int_0^{\xi} Q(\xi')dz', \; D^{\pm} = \pm(Q' + QP) \quad , \tag{55}$$

and substitute (52) for $\bar{q}$, w, p, ρ and η in (10) to (17). Next we collect the coefficients of $V(\beta^{\nu}\xi)$ and $V'(\beta^{\nu}\xi)$ and set them to zero. Then we multiply the coefficients of V' by $Q\beta^{-\nu+1}$ and add them together and subtract them from the corresponding coefficients of V. Making use of (54) and (55), and omitting the superscripts $\pm$, we obtain

$$\beta(\rho w - \rho_0\bar{q}\cdot\bar{k} + w_z) + \rho_t + \nabla\cdot(\rho_0\bar{q}) - D(\rho^{(2)}\xi_t + \rho_0\bar{q}^{(2)}\cdot\nabla\xi) = 0 \; . \tag{56}$$

$$\begin{aligned} &\beta(\rho_0\bar{q}_{\omega} - p\bar{k}) + \rho_0\bar{q}_t + \nabla p - D(\rho_0 q^{(2)}\xi_t + p^{(2)}\nabla\xi) \\ &= \beta\bar{q}_{zz} - (1/3)\bar{k}w_z + O(\beta^{-1}) \quad . \end{aligned} \tag{57}$$

$$p_3 + \rho = O(\beta^{-1}) \quad , \tag{58}$$

$$\rho^{\pm} = \rho_0'(p_0)p \quad ; \tag{59}$$

at $z = 0$,

$$q_z + O(\beta^{-2}) = 0, \; p - \eta + O(\beta^{-2}) = 0 \quad ,$$

$$\beta(\omega\eta - w) + \eta_t - D\xi_t \eta^{(2)} = 0 \quad , \tag{60}$$

at $z = -h$,

$$\bar{q}^{\pm} = 0, \; w^{\pm} = 0 \quad , \tag{61}$$

where $O(\beta^{-1})$, $O(\beta^{-2})$ denote terms of orders β^{-1} and β^{-2} respectively and will not be used in the sequel.

The equations for the first approximation are

$$\omega\rho_1 - \rho_0 \bar{k} \cdot \bar{q}_1 + (\rho_0 w_1)_z = 0 \quad , \tag{62}$$

$$\rho_0 \omega \bar{q}_1 - \bar{k} p_1 = \bar{q}_{1zz} \quad , \tag{63}$$

$$p_{1z} = -\rho_1 \quad , \tag{64}$$

$$\rho_1 = \rho_0'(p_0) p_1 \quad ; \tag{65}$$

at $z = 0$, $$\bar{q}_{1z} = 0, \; p_1 = \eta_1, \; \omega\eta_1 - w_1 = 0 \quad , \tag{66}$$

at $z = -h$, $$\bar{q}_1 = 0, \; w_1 = 0 \quad . \tag{67}$$

which are the same as (19) to (24). The results obtained for the first approximation in the ray method expansion can be carried over without change, and the Hamilton-Jacobi equations for $S^{\pm}$ are

$$\omega^{\pm} \rho_0(-h) = (k^{\pm})^2 \int_{-h}^{0} Q_1^{\pm} \, \rho_0 dz \quad , \tag{68}$$

the solutions of which determine two two-parameter families of rays

$$t^{\pm} = t^{\pm}(\sigma, \sigma_1, \sigma_2), \; \bar{r}^{\pm} = \bar{r}^{\pm}(\sigma, \sigma_1, \sigma_2) \quad . \tag{69}$$

To obtain a conservation relation for η_1, we need the equations for the second approximation

$$\omega\rho_2 + \rho_{1t} - \rho_0 \bar{q}_2 \cdot \bar{k} + \nabla \cdot (\rho_0 \bar{q}_1) + (\rho_0 w_2)_z - D(\rho_1^{(2)} \xi_t + \rho_0 \bar{q}^{(2)} \cdot \nabla \xi) = 0 \quad . \tag{70}$$

$$\rho_0 \omega \bar{q}_2 + \rho \cdot \bar{q}_{1t} - \bar{k} p_2 + \nabla p_1 - D(\rho_0 q_1^{(2)} \xi_t + p_1^{(2)} \nabla \xi) = \bar{q}_{2zz} - (1/3) \bar{k} W_{1z} \quad . \tag{71}$$

$$p_{2z} + \rho_2 = 0 \quad , \tag{72}$$

$$\rho_2 = \rho_0'(p_0)p_2 \quad ;$$

at $z = 0$,

$$q_{2z} = 0, \ p_2 = \eta_2 \quad , \tag{73}$$

$$\omega\eta_2 - w_2 + \eta_{1t} - D\xi_t\eta_1^{(2)} = 0 \quad ; \tag{74}$$

at $z = -h$, $$\bar{q}_2 = 0, \ w_2 = 0 \quad . \tag{75}$$

From (72) to (73), we easily obtain that

$$p_2 = \rho_0\eta_2, \ \rho_2 = \rho_0'(p_0)\rho_0\eta_2 \quad . \tag{76}$$

As seen from (71), (73) and (75), we may let

$$\bar{q}_2 = (\bar{k}\eta_2 - \nabla\eta_1)Q_1 + \bar{Q}_2 + \bar{Q}_3 \tag{77}$$

where Q_1, $\bar{Q}_2$ satisfy (27) to (29), and (45) to (47) respectively, and $\bar{Q}_3$ satisfies

$$\bar{Q}_{3zz} - \omega\rho_0\bar{Q}_3 = -D(\rho_0 q_1^{(2)}\xi_t + p_1^{(2)}\nabla\xi) \quad , \tag{78}$$

$$\bar{Q}_{3z} = 0 \quad \text{at} \quad z = 0 \quad , \tag{79}$$

$$\bar{Q}_3 = 0 \quad \text{at} \quad z = -h \quad . \tag{80}$$

We now integrate (70) with respect to z from $z = -h$ to $z = 0$, and make use of (27), (68), (74) and (75) to obtain

$$\begin{aligned} &\int_{-h}^{0} [\eta_{1t}\rho_0'(p_0)\rho_0 + \rho_0 Q_1\bar{k}\cdot\nabla\eta_1 - \rho_0\bar{k}\cdot\bar{Q}_2 + \nabla\cdot(\rho_0\bar{k}\eta_1 Q_1)]dz + \eta_{1t} \\ &= \int_{-h}^{0} \rho_0\bar{k}\cdot\bar{Q}_3 dz + \int_{-h}^{0} D(\rho_1^{(2)}\xi_t + \rho_0\bar{q}_1^{(2)}\cdot\nabla\xi)dz + D\xi_t\eta_1^{(2)} \quad . \end{aligned} \tag{81}$$

Note that the left hand side of (81) is the same as (48). We multiply both sides of (81) by $2\eta_1$, and follow the same derivation given in Appendix A to obtain

$$\begin{aligned} &(I\eta_1^2)_t + \nabla\cdot 2\bar{k}\int_{-h}^{0} \eta_1^2\rho_0 Q_1 dz - A\eta_1^2 \\ &= \int_{-h}^{0} 2\eta_1\rho_0\bar{k}\cdot\bar{Q}_3 dz + 2\eta_1\int_{-h}^{0} D(\rho_1^{(2)}\xi_t + \rho_0\bar{q}_1^{(2)}\cdot\nabla\xi)dz + 2\eta_1 D\xi_t\eta_1^{(2)} \quad . \end{aligned} \tag{82}$$

In Appendix 2, we show that if

$$\eta_1 = R(\xi)\zeta_1, \quad \text{and} \quad 2R' = RDQ^{-1} \quad , \tag{83}$$

Then ζ_1 satisfies the same conservation relation in the ray method expansion, and it follows that, along a ray

$$J^{\pm}I^{\pm}R^{-2}(\eta_1^{\pm})^2 \exp[-\int_{t_0}^{t} A^{\pm}(I^{\pm})^{-1}dt] = \text{constant} \quad . \tag{84}$$

From (55) and (79), we find that

$$R = Q^{1/2} \exp[(1/2) \int^{\xi} P(\xi')d\xi'], \; P = [\log (R^2/Q)]' \quad . \tag{85}$$

At an anomaly, we may choose R so that $R^{-1}\eta_1^{\pm}$ tend to a finite limit there. P, Q and μ should be suitably chosen to make (53) as simple as possible.

4. Discussion

To apply the present results to flow of a compressible viscous fluid with a fixed upper boundary, we simply replace the boundary conditions at $z = 0$ by $\bar{q} = 0$, $w = 0$, at $z = H$, the equation of the upper boundary, and η_1 by $p_1(H)$, the value of p_1 at $z = H$. The Hamilton-Jacobi equation assumes the form

$$\omega(\rho_0(-h) - \rho_0(H)) = k^2 \int_{-h}^{0} Q_1\rho_0 dz \quad ,$$

where Q_1 satisfies (28) with $Q_1 = 0$ at $z = -h, H$. The conservation relation along a ray is still the same as (50) except I and A are now given by

$$I = \rho_0(-h) - \rho_0(H) - k^2 \int_{-h}^{H} Q_{1\omega}\rho_0 dz \quad ,$$

$$A = k^2 \int_{-h}^{H} \rho_0(z) \int_{-h}^{H} G(z,z',\omega,h,H)(\omega\rho_{0z'} + k^2 Q_1(z')\rho_0(z'))dz' \quad .$$

The uniform asymptotic expansion for this case can also be dealt with accordingly.

As an illustration of the construction of a uniform asymptotic expansion, we consider a strictly convex caustic where $J(t) = 0$. Following [9], we choose $\mu = 2/3$, $Q = \xi^{1/2}$, and $R = \xi^{1/4}$. From (85), $P = 0$ and (53) becomes the Airy equation

$$V''(\beta^{2/3}\xi) - \beta^{2/3}\xi\, V(\beta^{2/3}\xi) = 0 \quad .$$

If there are two caustics corresponding to $\xi = 0$, $\xi = \xi_1$, then (53) becomes the equation for Weber functions

$$V''(\xi) - \beta^2\xi(\xi_1-\xi)V''(\xi) = 0 \quad ,$$

where $\nu = 0$, $R = \xi^{1/4}(\xi_1-\xi)^{1/4}$, $P = 0$ and $W = R^2$. The contruction of a uniform asymptotic expansion at a line of zero depth may follow the procedure developed in [12], and we shall omit the derivation here.

Finally we make some remarks regarding the possible extensions and applications of the ray method. If the fluid region is channel-like, that is, the rigid bottom varies slowing in one direction only, say the x^* direction. In this case, we introduce the following nondimensional variables

$$(x,y,z) = (\beta^{-2}x^*,y^*,z^*)/H \quad ,$$
$$\bar{q} = (v,w) = \beta(v^*,w^*)/(gH)^{1/2} \quad ,$$

and the remaining ones are the same as before. Then we assume $S = S(t,x)$ and the ray method can be carried out without much difficulty.

In this paper we only consider the case of large Reynolds number R. With a change of the stretched variables, the ray method can also be extended to the case of finite R, but the derivations will be much more involved. We also note that the results obtained are only valid for positive phase function S and then the nonlinear terms omitted will have a negligible effect. One of the interesting applications of the method to physical problems is to consider a point source on the free surface, for example to model a well in an unconfined aquifer. Mathematically, we simply add a Dirac distribution on the right-hand side of (17). First we construct the rays from the source based upon (33) and (34). The phase and amplitude functions are given by (36) and (51). Then the initial conditions for them are determined by the asymptotic expansion of the exact solution of a canonical problem based upon a system of boundry layer equations near the origin. The procedure to solve a similar problem may be found in [11].

Appendix A

In (48), we first note that

$$2\eta_1 \int_{-h}^{0} \rho_0 \Omega_1 \bar{k}\cdot\nabla\eta_1 + 2 \int_{-h}^{0} \eta_1 \nabla\cdot(\rho_0 \bar{k}\eta_1\Omega_1)dz$$

$$= 2\nabla\cdot\int_{-h}^{0} \bar{k}\eta_1^2\rho_0\Omega_1 dz \quad ,$$

where $\Omega_1 = 0$ at $z = -h$. Next from (27), (45) to (47),

$$\bar{\Omega}_2 = \int_{-h}^{0} G(z,z',\omega,h)[\rho_0(z')(\bar{k}\eta_1\Omega_1)_t + (1/3)\bar{k}w_{1z}]dz \quad .$$

We collect the terms with time derivatives in (48) to obtain

$$2\eta_1\eta_{1t}[\int_{-h}^{0} \rho_0'(P_0)\rho_0 dz + 1] - 2\eta_1 \int_{-h}^{0} \rho_0(z) \int_{-h}^{0} \bar{k}\cdot\rho_0(z')\eta_1\Omega_1(\zeta'))_t \times$$
$$G(z,z',\omega,h)dz'dz$$
$$= (\eta_1^2)_t\ \rho_0(-h) + 2\int_{-h}^{0} \bar{k}\cdot(\bar{k}\eta_1\Omega_1)_t\ \eta_1\Omega_1(z')\rho_0(z')dz'$$
$$= \frac{\partial}{\partial t}\ [\eta_1^2\rho_0(-h) + \eta_1^2 \int_{-h}^{0} k^2\Omega_1^2\rho_0 dz'] \quad .$$

Where we have changed the order of integration and made use of the symmetry property of the Green's function in the second integral. Hence, (48) assumes the form

$$\frac{\partial}{\partial t}\ \{\eta_1^2[\rho_0(-h) + \int_{-h}^{0} k^2\Omega_1^2\rho_0 dz'\} + 2\nabla\cdot\int_{-h}^{0} \bar{k}\eta_1^2\rho_0\Omega_1 dz$$
$$- A\eta_1^2 = 0 \quad . \tag{A.1}$$

Finally we go back to (33). Assuming ω as a function of $\bar{k}$, we have, by differentiating both sides of (33) with respect to k_i where $\bar{k} = (k_1,k_2)$.

$$\partial\omega/\partial k_i = dx_i/dt = 2k_i \int_{-h}^{0} \rho_0\Omega_1 dz[\rho_0(-h)-k^2 \int_{-h}^{0} \Omega_{1\omega}\rho_0 dz]^{-1} \quad . \tag{A.2}$$

We show that

$$-\int_{-h}^{0} \rho_0\Omega_1^2 dz = \int_{-h}^{0} \rho_0\Omega_{1\omega}dz \quad . \tag{A.3}$$

From (28) to (30), we find that $\Omega_{1\omega}$ satisfies

$$\Omega_{1\omega zz} - \rho_0\omega\Omega_{1\omega} = \rho_0\Omega_1 \quad ,$$
$$\Omega_{1\omega z} = 0 \quad , \quad \text{at} \quad z = 0 \quad ,$$
$$\Omega_{1\omega} = 0 \quad , \quad \text{at} \quad z = -h \quad .$$

Hence,

$$\Omega_{1\omega} = \int_{-h}^{0} G(z,z',\omega,h)\rho_0\Omega_1 dz' \quad . \tag{A.4}$$

It follows that

$$\int_{-h}^{0} \rho_0(z)\Omega_{1\omega}(z)dz = \int_{-h}^{0} \rho_0(z) \int_{-h}^{0} G(z,z',\omega,h)\rho_0(z')\Omega_1(z')dz'$$

$$= - \int_{-h}^{0} \rho_0\Omega_1^2 dz$$

by integration by parts, symmetry of the Green's function and (31). From (A.1) to (A.3), (49) follows.

Appendix B

We substitute $\eta_1 = R(\xi)\zeta$, in (82) and obtain

$$R^2[(I\zeta_1^2)_t + 2\nabla\cdot\bar{k}\int_{-h}^{0}\rho_0\Omega_1\zeta_1^2 dz - Az_1^2] = -2RR'[I\zeta_1^2\xi_t$$

$$+ 2\nabla\xi\cdot\bar{k}\int_{-h}^{0}\zeta_1^2\rho_0\Omega_1 dz] + R^2\int_{-h}^{0} 2\zeta_1[\rho_0\bar{k}\cdot\bar{\Omega}_3 dz + \tag{B.1}$$

$$+ D(P_1^{(2)}\xi_t + \rho_0\bar{q}_1^{(2)}\cdot\nabla\xi) + D\xi_t\zeta_1^{(2)}]dz = \text{RHS} \quad ,$$

where $\zeta_1^{(2)} = (\zeta_1^+ - \zeta_1^-)/(2Q)$. If we can show that the RHS of (B.1) is zero under (83), then ζ_1 satisfies the same transport equation as (49), and (84) follows.

From (26), (27) and (54),

$$\nabla\xi = (\bar{k}^- - \bar{k}^+)/(2Q), \ \xi_t = (\omega^+ - \omega^-)/(2Q) \quad ,$$

$$P_1^{(2)} = \rho_0(\eta_1^+ - \eta_1^-)/(2Q), \ \rho_1^{(2)} = \rho_0'\rho_0(\eta_1^+ - \eta_1^-)/(2Q) \quad ,$$

$$\bar{q}_1^{(2)} = (\bar{k}^+\eta_1^+\Omega_1^+ - \bar{k}^-\eta_1^-\Omega_1^-)/(2Q) \quad .$$

We substitute them in the RHS of (B.1) and consider the upper sign first. The coefficient of $(\zeta^+)^2$ is

$$(\omega^+ - \omega^-)I^+/2 + (\bar{k}^- - \bar{k}^+)\cdot\bar{k}^+\int_{-h}^{0}\rho_0\Omega_1^+ dz$$

which can easily be shown to be zero by use of (31) and (A.4). The coefficient of $\zeta^+\zeta^-$ is found as

$$\bar{k}^+\cdot\bar{k}^-\{(\omega^+-\omega^-)\int_{-h}^{0}\rho_0(z)\int_{-h}^{0}\rho_0(z')G^+(z,z',\omega^+,h)\Omega_1^-(z')dz'dz$$

$$+ \int_{-h}^{0}\rho_0(z)\int_{-h}^{0}\rho_0(z')[G^+(z,z',\omega^+,h) - G^-(z,z',\omega^-,h)]dz'dz\} \quad , \tag{B.2}$$

where we have used the relations

$$\omega^\pm\rho_0(-h) = (k^\pm)\int_{-h}^{0}\rho_0\Omega_1^\pm dz \quad .$$

To show (B.2) is zero, we apply (28) and (30) for $\Omega_1^\pm$:

$$L^\pm\Omega_1^\pm = \Omega_{1zz}^\pm - \rho_0\omega^\pm\Omega_1^\pm = -\rho_0 \quad , \tag{B.3}$$

$$\Omega_{1z}^\pm = 0 \ \text{ at } \ z = 0 \quad , \tag{B.4}$$

$$\Omega_1^{\pm} = 0 \quad \text{at} \quad z = -h \quad . \tag{B.5}$$

Then let $\int_{-h}^{0} UVdz = (U,V)$, and we have from (3)

$$\begin{aligned} &(\Omega_1^-, L^+\Omega_1^+) - (\Omega_1^+, L^-\Omega_1^-) + (\Omega_1^-, -\rho_0\omega^+\Omega_1^+) \\ &- (\Omega_1^+, -\rho_0\omega^-\Omega_1^-) = (\Omega_1^-, -\rho_0) - (\Omega_1^+, -\rho_0) \quad . \end{aligned} \tag{B.6}$$

By the boundary conditions (B.3) and (B.4), (B.5) becomes

$$(\omega^+ - \omega^-)(\rho_0\Omega_1^+, \Omega_1^-) + (\rho_0, \Omega_1^+) - (\rho_0, \Omega_1^-) = 0 \quad .$$

The LHS of the above equation reduces to (B.2) by integration by parts, symmetry of the Green's function and (31). Similarly we can also show that the RHS of (B.1) coresponding to the lower sign is also equal to zero.

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An asymptotic method is developed for the solution of linearized equations governing the flow of a compressible viscous fluid with a free surface. The approach used here is based upon the ray method expansion originally developed by Keller. A general uniform asymptotic expansion is also constructed to remove anomalies in which an amplitude function in the ray method expansion becomes infinite.

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